Graphical continuous Lyapunov models

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Based on joint works with Carlos Améndola, Mathias Drton,
Benjamin Hollering, Sarah Lumpp, Pratik Misra, Daniela Schkoda and Liam Solus

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Causal modeling with directed graphs

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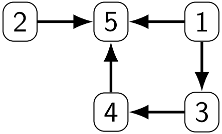

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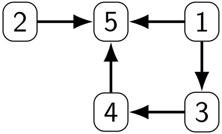

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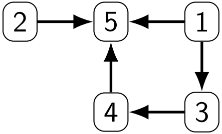

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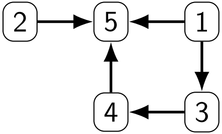

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- Solutions to this system are also Gaussian with covariance matrix satisfying the congruence equation $(I - \Lambda)^{\mathsf{T}} \Sigma (I - \Lambda) = \Omega.$

Reasoning with graphical models

AGU PUBLICATIONS

Water Resources Research

RESEARCH ARTICLE

10.1002/2017WR020412

Key Points:

- We develop a statistical graphical model to characterize the statewide California reservoir system
- We quantify the influence of external physical and economic factors (e.g., statewide PDSI and consumer price index) on the reservoir network
- Further analysis gives a system-wide health diagnosis as a function of PDSI, indicating when heavy management practices may be needed

Supporting Information:

- Supporting Information S1
- Supporting Information S2

A Statistical Graphical Model of the California Reservoir System

A. Taeb[1], J. T. Reager[2], M. Turmon[2], and V. Chandrasekaran[3]

[1]Department of Electrical Engineering, California Institute of Technology, Pasadena, CA, USA, [2]Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA, USA, [3]Department of Computing and Mathematical Sciences and Department of Electrical Engineering, California Institute of Technology, Pasadena, CA, USA

Abstract The recent California drought has highlighted the potential vulnerability of the state's water management infrastructure to multiyear dry intervals. Due to the high complexity of the network, dynamic storage changes in California reservoirs on a state-wide scale have previously been difficult to model using either traditional statistical or physical approaches. Indeed, although there is a significant line of research on exploring models for single (or a small number of) reservoirs, these approaches are not amenable to a system-wide modeling of the California reservoir network due to the spatial and hydrological heterogeneities of the system. In this work, we develop a state-wide statistical graphical model to characterize the dependencies among a collection of 55 major California reservoirs across the state; this model is defined with respect to a graph in which the nodes index reservoirs and the edges specify the

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  - Markov equivalence = ambiguity about the direction of causality.

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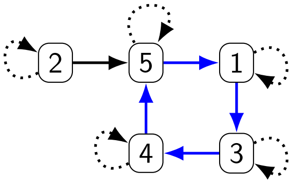

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$$\mathrm{d}\mathbb{X}(t) = M(\mathbb{X}(t) - \mu)\mathrm{d}t + D\mathrm{d}\mathbb{W}(t),$$

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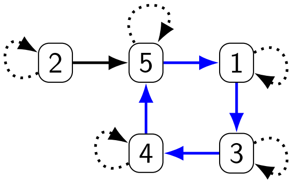

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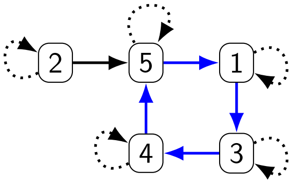

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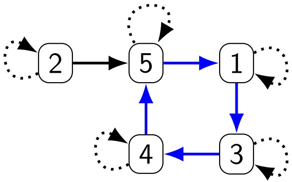

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- The stationary distribution is Gaussian with covariance matrix satisfying the Lyapunov equation $M\Sigma + \Sigma M^{T} + DD^{T} = 0.$

Parametrization

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⋛ The Lyapunov model is an irreducible algebraic variety!* ⋚

*Actually a semialgebraic set with irreducible Zariski closure.

Parameter identifiability

$$M\Sigma + \Sigma M^{\mathsf{T}} + C = 0.$$

- The Lyapunov equation is **also** a linear matrix equation in M, equivalent to:

$$(\Sigma \otimes I + (I \otimes \Sigma)K_n)\operatorname{vec} M = -\operatorname{vec} C,$$

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Theorem ([Det+23])

Let G be a simple directed graph (i.e., having no directed 2-cycles). Given Σ in the graphical continuous Lyapunov model of G with fixed diffusion matrix C, the parameter matrix M is uniquely recoverable as a rational function of Σ.

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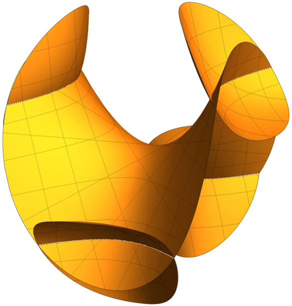

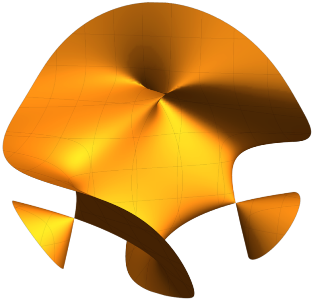

Restricted trek rule

Lemma ([Boe+24])

Let $G = (V, E)$ be a DAG, fix $C = 2I$ and $m_{ii} = -1/2\zeta$ for all $i \in V$. Then the following trek rule holds:

$$\sigma_{ij} = \sum_{\substack{T:(\ell,r)\text{-trek} \\ \text{from } i \text{ to } j}} 2\zeta^{\ell+r+1} \binom{\ell+r}{\ell} m_T, \tag{1}$$

where the trek monomial m_T associated to a trek T is given by $m_T = \prod_{e \in T} m_e$, i.e., the product over all the edges e in that trek.

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- For acyclic linear SEMs, a similar trek rule holds but without the blue term.
- This disturbs the conditional independence structure familiar from linear SEMs.

Conditional independence

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- All conditional independence statements are implied by absences of treks.
- Lyapunov models are **not** defined by conditional independence relations.

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$$\sigma_{11}\sigma_{12}^2\sigma_{13}\sigma_{22} - \sigma_{11}^2\sigma_{13}\sigma_{22}^2 - \sigma_{11}\sigma_{12}^3\sigma_{23} + \sigma_{11}\sigma_{12}\sigma_{13}^2\sigma_{23} + \sigma_{11}^2\sigma_{12}\sigma_{22}\sigma_{23} + \sigma_{12}\sigma_{13}^2\sigma_{22}\sigma_{23} - \sigma_{11}^2\sigma_{13}\sigma_{23}^2 - 2\sigma_{12}^2\sigma_{13}\sigma_{23}^2 + \sigma_{11}\sigma_{13}\sigma_{22}\sigma_{23}^2 - \sigma_{11}\sigma_{12}^2\sigma_{13}\sigma_{33} - \sigma_{11}\sigma_{13}\sigma_{22}^2\sigma_{33} + \sigma_{11}^2\sigma_{12}\sigma_{23}\sigma_{33} + \sigma_{12}^3\sigma_{23}\sigma_{33} = 0$$

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- On the space of correlation matrices $\sigma_{11} = \sigma_{22} = \sigma_{33} = 1$, this turns reducible

$$(\sigma_{13} - \sigma_{12}\sigma_{23}) \cdot (1 - \sigma_{12}\sigma_{13}\sigma_{23}) = 0$$

and implies the CI relation $[1 \perp\!\!\!\perp 3 \mid 2]$ which does not hold on the entire model.

What we know so far

		Rat. param.	Rat. ident.	Smooth	CI Markov prop.	Struct. ident.
LSEM	Acyclic	✓	✓	✓	✓	✗
	Simple	✓	✗	?	✓	✗
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- Relation of a graph's linear SEM and Lyapunov model for correlation matrices?

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