Entropy profiles and algebraic matroids

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arXiv:2502.20355

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Discrete Mathematics & Geometry seminar,
TU Berlin, 11 June 2025

Supported by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 101110545.

Funded by
the European Union

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- Entropy profiles are "rank functions": monotone and submodular.

Entropy as information

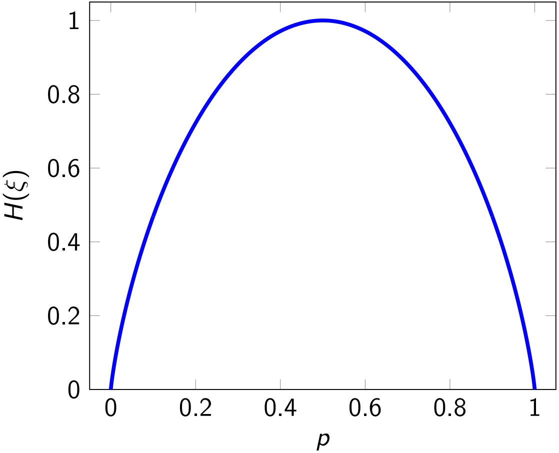

Figure: Entropy of a binary random variable ξ as a function of $p = p(\xi = \text{heads})$.

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- Graphical models in statistics and causality are defined by CI assumptions (e.g., Bayesian networks and d-separation in graphs).
- Cryptographic protocols use FD and CI constraints to specify operation and information-theoretic security (e.g., secret sharing).
- Quantities in information theory are defined by linear optimization over entropy profiles with FD and CI constraints (e.g., common information).

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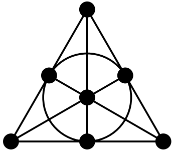

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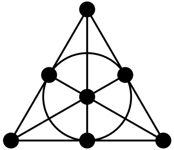

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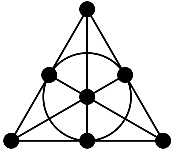

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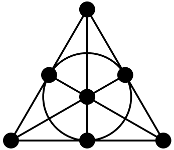

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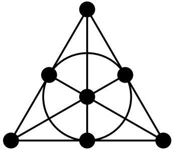

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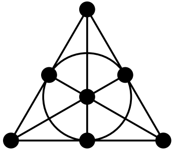

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- The optimal information ratio $\sigma(\mathscr{Q}) = \inf \{\sigma(h) : h \models \mathscr{Q}\}$ can be determined by linear optimization over the set of all entropy profiles satisfying linear conditions.

The entropy region and information inequalities

Let $H_N^* \subseteq \mathbb{R}^{2^N}$ consist of all h_ξ where ξ is an N-variate discrete random vector. H_N^* is the image of $\bigcup_{d_1=1}^{\infty} \cdots \bigcup_{d_n=1}^{\infty} \Delta(d_1, \ldots, d_n)$ under the transcendental map $\xi \mapsto h_\xi$.

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Theorem ([Mat07a])

$\overline{H_N^*}$ *is not polyhedral for* $|N| \geq 4$.

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- Kaced and Romashchenko [KR13] proved that (1) is **essentially conditional**, i.e., there are no $\lambda, \mu \in \mathbb{R}$ such that (2) holds.

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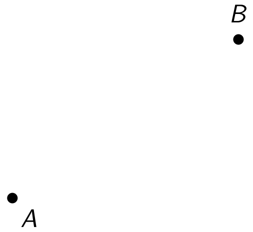

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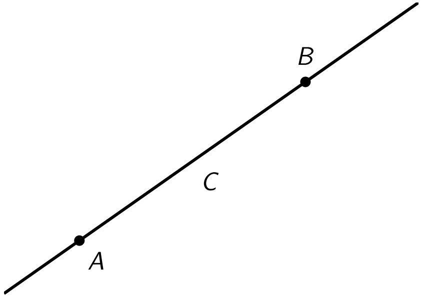

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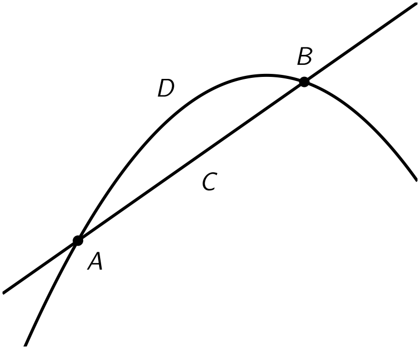

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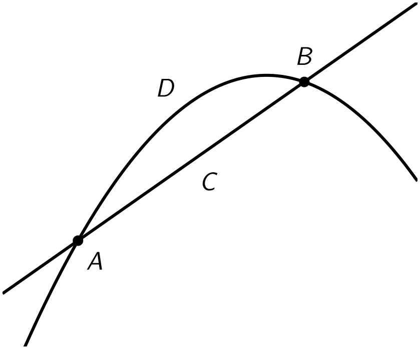

The coordinates of (A, B, C, D) define the support of a distribution on $\mathbb{F}_q^2 \times \mathbb{F}_q^2 \times \mathbb{F}_q^2 \times \mathbb{F}_q^3$ and the distribution is uniform on this set.

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- Elementary parameter counting yields

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- However, $H(C:D) = \log(q) - \log(q-1) + \log(2)$. The $\log(2)$ term reflects that only half of all pairs (C, D) defined over $\mathbb{F}_q$ intersect in two $\mathbb{F}_q$-rational points!

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which becomes negative for any λ, μ as $q \to \infty$.

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Can this be done by computer algebra?

Model theory of finite fields

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Consider a formula $\varphi(x_1, \dots, x_n, y_1, \dots, y_m)$. There exist finitely many formulas $\psi_k(y_1, \dots, y_m)$, indexed by $k \in K$, with accompanying $\mu_k \in \mathbb{Q}$ and $d_k \in \mathbb{N}$ such that

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- *There exists a unique $k \in K$ such that $\mathbb{F}_q \models \psi_k(b)$.*

Model theory of finite fields

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- More briefly: the smallest set of sets containing all varieties defined over $\mathbb{F}_q$ and closed under complement and projection.

Theorem ([CDM92])

Consider a formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$. There exist finitely many formulas $\psi_k(y_1, \ldots, y_m)$, indexed by $k \in K$, with accompanying $\mu_k \in \mathbb{Q}$ and $d_k \in \mathbb{N}$ such that for every sufficiently large finite field $\mathbb{F}_q$ and every $b \in \mathbb{F}_q^m$:

- *There exists a unique $k \in K$ such that $\mathbb{F}_q \models \psi_k(b)$.*
- *$\mathbb{F}_q \models \psi_k(b)$ if and only if $|\varphi(\mathbb{F}_q^n; b)| = \mu_k q^{d_k} + \mathcal{O}(\mu_k q^{d_k - 1/2})$.*

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Theorem ([FHJ94])

Fiber decompositions are computable. Moreover, one can compute a bound $m \in \mathbb{N}$, numbers $d_k \in \mathbb{N}$ and non-negative $\mu_k \in \mathbb{Q}$ such that for every finite extension $\mathbb{G}/\mathbb{F}$:

$$|X(\mathbb{G})| = \mu_k |\mathbb{G}|^{d_k} + \mathcal{O}(\mu_k |\mathbb{G}|^{d_k - 1/2}), \text{ where } k \equiv [\mathbb{G} : \mathbb{F}] \pmod{m}.$$

Computability of the entropy profiles

Theorem

Let X be an $\mathbb{F}$-definable set in n free variables and $\xi(\mathbb{G})$ the uniform distribution on $X(\mathbb{G})$. For a projection $\pi_I(X)$ let $(Y_k : k \in K)$, be a fiber decomposition and set $X_k = X \cap \pi_I^{-1}(Y_k)$. For large enough $\mathbb{G}/\mathbb{F}$, the entropy profile satisfies

$$h_{\xi(\mathbb{G})}(I) = \sum_{\dim_{\mathbb{G}}(X_k)=\dim_{\mathbb{G}}(X)} \frac{\mu_{\mathbb{G}}(X_k)}{\mu_{\mathbb{G}}(X)} \log\left(\frac{\mu_{\mathbb{G}}(X)\mu_{\mathbb{G}}(Y_k)}{\mu_{\mathbb{G}}(X_k)}|\mathbb{G}|^{\dim_{\mathbb{G}}(Y_k)}\right) + \mathcal{O}\left(\frac{\log|\mathbb{G}|}{\sqrt{|\mathbb{G}|}}\right).$$

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- The sequence $\left(\frac{1}{\log|\mathbb{G}|} h_{\xi(\mathbb{G})} : \mathbb{G} \supseteq \mathbb{F}\right)$ has finitely many convergent subsequences and their (rational!) limits can all be computed.

Algebraic matroids

Theorem

Moreover, if X is an $\mathbb{F}$-irreducible algebraic variety, then there exists a tower of finite fields $\mathbb{F} = \mathbb{G}_0 \subseteq \mathbb{G}_1 \subseteq \dots$ with

$$\lim_{n\to\infty} \frac{1}{\log|\mathbb{G}_n|} h_{\xi(\mathbb{G}_n)}(I) = \dim \pi_I(X(\overline{\mathbb{F}})), \quad \textit{for every } I \subseteq N.$$

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- Entropy profile can be seen as a "valuated" refinement of the algebraic matroid.

Example using Galois stratification

- To eliminate x from the variety defined by $x^3 + ax^2 + bx + c = 0$, stratify the triples (a, b, c) according to the number of rational roots of $f(a, b, c) \in \mathbb{F}[x]$.

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- The Chebotarev density theorem computes the density of triples with given conjugacy class $\mathcal{C}$:

$$\frac{|\mathcal{C}|}{[\Omega : \mathbb{F}(a, b, c)]} = \frac{|\mathcal{C}|}{6},$$

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Splitting type	$[1, 1, 1]$	$[1, 2]$	$[3]$	$[1, 1^2]$	$[1^3]$
Conjugacy class	id	$(1\ 2)$	$(1\ 2\ 3)$	–	–
Density	$1/6$	$3/6$	$2/6$	0	0
Rational roots	3	1	0	2	1

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