Tobias Boege

Two universality results for Gaussian CI models

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Gaussian conditional independence

Consider random variables $(\xi_i)_{i\in \mathbb{N}} \sim \mathcal{N}(\mu, \Sigma)$. The conditional independence (CI) statement $\xi_i \perp \!\!\! \perp \xi_j \mid \xi_K$ conveys, informally, that if ξ_K is known, then learning the value of one variable does not give any information about the other one.

Definition

The polynomial $\Sigma[K] := \det \Sigma_{K,K}$ is a *principal minor* of Σ and $\Sigma[ij|K] := \det \Sigma_{iK,jK}$ is an *almost-principal minor*.

If Σ is positive-definite, then $\Sigma[K] > 0$, and $\xi_i \perp \xi_j \mid \xi_K$ holds if and only if $\Sigma[ij \mid K] = 0$.



Almost-principal minors

$$\begin{split} & \Sigma[ij|k] = x_{ij} \\ & \Sigma[ij|k] = x_{ij} x_{kk} - x_{ik} x_{jk} \\ & \Sigma[ij|kl] = x_{ij} x_{kk} x_{ll} - x_{il} x_{jl} x_{kk} + x_{il} x_{jk} x_{kl} + x_{ik} x_{jl} x_{kl} - x_{ij} x_{kl}^2 - x_{ik} x_{jk} x_{ll} \\ & \Sigma[ij|klm] = x_{ij} x_{kk} x_{ll} x_{mm} + x_{im} x_{jm} x_{kl}^2 - x_{im} x_{jl} x_{kl} x_{km} - x_{il} x_{jm} x_{kl} x_{km} + x_{il} x_{jl} x_{km}^2 \\ & - x_{im} x_{jm} x_{kk} x_{ll} + x_{im} x_{jk} x_{km} x_{ll} + x_{ik} x_{jm} x_{km} x_{ll} - x_{ij} x_{km}^2 x_{ll} \\ & + x_{im} x_{jl} x_{kk} x_{lm} + x_{il} x_{jm} x_{kk} x_{lm} - x_{im} x_{jk} x_{kl} x_{lm} - x_{ik} x_{jm} x_{kl} x_{lm} \\ & - x_{il} x_{jk} x_{km} x_{lm} - x_{ik} x_{jl} x_{km} x_{lm} + 2 x_{ij} x_{kl} x_{km} x_{lm} + x_{ik} x_{jl} x_{kl} x_{mm} \\ & - x_{ij} x_{kk} x_{lm}^2 - x_{il} x_{jl} x_{kk} x_{mm} + x_{il} x_{jk} x_{kl} x_{mm} + x_{ik} x_{jl} x_{kl} x_{mm} \\ & - x_{ij} x_{kl}^2 x_{mm} - x_{ik} x_{jk} x_{ll} x_{mm} \\ & \vdots \end{split}$$



Models and inference

Definition

A CI constraint is a CI statement $\xi_i \perp \!\!\! \perp \xi_j \mid \xi_K$ or its negation $\neg(\xi_i \perp \!\!\! \perp \xi_j \mid \xi_K)$. They are algebraic conditions on the entries of Σ , equivalent to vanishing or non-vanishing of the almost-principal minors $\Sigma[ij|K]$.

Definition

The *model* of a set of CI constraints is the set of all positive-definite matrices which satisfy the constraints.



Models and inference

Consider two sets of CI statements \mathcal{L} and \mathcal{M} :

Reasoning about relevance statements in normally distributed random variables is the same as reasoning about the vanishing of very special kinds of determinants in the positive-definite matrices.



Model M1	Model M2	Model M3	Model M4	Model M5	Model M6	Model M49	Model M50	Model M51	Model M52	Model M53	
	2 41-32					4000	4 0 -3 0	4000			
0 4 1 -				0 4 2 -3		0 4 1-2	0401	0 4 0 -3	0100	-3 4 1 1	
2 1 4 - a -2 -3 -3				2 2 4 -3		0 1 4 -2	3 9 4 9	0.3 0 4	0011	-3 1 4 3 e e -3 1 3 4	
		-		-	\sim						
Model M7	Model M8	Model M9	Model M10	Model M11	Model M12	Model M54	Model M55	Model M56	Model M57	Model M58	
4 0 -2 0 4 -3		9 4 2 -3 3 2 4 -3 1						· ·	•		
-2 -3 4						r^	1	1		{ }	
	4 - 2 2 -1 4				0 1-2 0 4			<u> </u>	a V a	()	
Model M13	Model M14	Model M15	Model M16	Model M17	Model M18	Model M59	Model M60	Model M61	Model M62	Model M63	Model M64
0 4-12	2 0 4 1 2 -2	9 6 1 -4 2	4 0 2 -1	9 4 0 -2 1		4 - 3 - 2 1	2 0 1 -1	◆ 5 0 4 ⋅3	8-2-4-7	10 0 4 -6	4 1 1 -2
-1 4 -2						-3 4 2 -2	0 2 -1 -1	052-4	-2 8 4 -2	0 10 -3 -8	1 4 - 2 - 2
2 -2 4						-2 2 4 2	1-12-1	4 2 5 -4	-4 4 8 2	4-310 0	1.2 4.2
e e -2 2 -3	4 e e -2 -2 -2 -	e 23-3 (6 6 1334	0 12-24	B 2-1-1 4	e e 1-2 2 4		-3-4-45	0 -7-228	-6-8 0 10	6 -2-2-2 4
Model M19	Model M20	Model M21	Model M22	Model M23	Model M24	Model M65	Model M66	Model M67	Model M68	Model M69	Model M70
402						9 0 -6 -6		14-13-11-7	10 0 -6 3	25 0 20 -15	1 2 1 1 · 1
2-14						0 9 3 -3	1 / 1 44 48 80 44 1	13 4 7 4 3	8-810-5	/20 15 25 24	1 1/2004 / 4 4 4 4
-2-3 0			2 2 -1 4			-6-3-19	35 -60 -64 80	7 2 13 14	a 3 4 -5 10		-1-1-2 2
Model M25	Model M26	Model M27	Model M28	Model M29	Model M30	Model M71	Model M72	Model M73	Model M74	Model M75	-
Model M25		Model M27	Model M28	Model M29	Model M30	Model M71	Model M72	Model M73	Model M74	Model M75	Model M76
Model M25	3 0 8-3-6-4	9 804-	0 10 -4 -5 -8	0 10 -4 -8 -5	4 0 -2 -2		10 0 -6 0	A 21-11	201-1	4 1 2 -1	Model M76
804	3 8-3-6-6	8 0 4 -2	10-4-5-8	10 -4 -8 -5	4 0 -2 -2 0 4 1 -1	5 0 -3 4 0 5 -4 -3 -3 -4 5 0	10 0 -6 0 0 10 -8 5 -6 -8 10 -4	2 1 -1 1 1 2 1 -1 1 1 2 -2	2 0 1 -1 0 2 -1 1 1 -1 2 -2	4 1 2 ·1 1 4 2 ·4 2 2 4 ·2	Model M76 5 0 3-3 0 5-4 4 3-5 5-5
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8 0 4 4 0 8 4 4 4 5 5 6 7 7 7 8 7 8 7 8 7 8 7 8 7 8 7 8 7 8 7	3 8 3 64 6 8 6 6 4 2 8 6 6 8 7 6 4 2 8 7 6 7 6 7 6 7 6 7 6 7 6 7 6 7 6 7 6 7	Model M45	Model M40 Model M40-21 M60-21 M60-2	10.4.8.4.10.2.8 4.10.2.8 8.2.10.4 8.2.10.4 Model M35 4.0.2.3 0.4.3.2.0.4 Model M41 12.3.6.8 12.9.6 6.6.912.8 6.6.912.8 10.4.10.4 10.4.10	4 0 2 2 2 4 4 1 2 2 2 4 4 1 2 2 2 4 4 1 2 2 2 4 4 1 2 2 2 4 4 4 0 2 0 0 0 1 4 4 0 4 0 0 0 0 4 4 0 4 0 0 0 0	0 0 3 4 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	10 0 -6 0 0 10 -8 5 0 10 -8 5 8 -10 -4 0 5 -4 10 Model M78 4-1 2-2 1 4 -2 2-2 4 4 2 2 2 4 4 Model M84 1 0 0 0 0 1 1 1 0 1 1 1 0 1 1 1	2 1-1 1 1 2 1-1 1 1 2 1-1 1 1 2 1-1 1 1 2 1 2	2 0 1 1 1 0 2 -1 1 1 1 2 -2 1 1 1 2 -2 1 1 1 2 -2 1 1 1 2 2 2 1 1 1 2 2 2 1 1 1 2 2 2 1 2 1 1 2 1	4 1 2 -1 1 4 2 -4 2 2 4 -2 -1 -4 -2 4 Model MB1 2 -2 -1 -2 -2 2 1 2 -1 1 2 1 2	Model M76 5 0 3 -3 0 5 -4 4 3 -5 5 5 2 4 -5 5 Model M82 2 0 0 0 0 2 -2 1 0 -2 2 -1
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Petr Šimeček. Gaussian representation of independence models over four random variables. In *COMPSTAT conference*, 2006.



Model M85





1 a b c
$$a = \frac{3}{632836}\sqrt{1107463}$$
, a 1 d e $b = 10c = \frac{100}{158209}\sqrt{1107463}$ b d 1 f c e f 1 $d = 10e = \frac{3}{4}$, $f = \frac{1}{10}$



Šimeček's Question

Does every non-empty Gaussian CI model contain a rational point?

Model M85



1	а	b	С	$a = \frac{3}{632836} \sqrt{1107463},$
a	1	d	е	$b = 10c = \frac{100}{158209} \sqrt{1107463}$
b	d	1	f	158209
С	е	f	1	$d = 10e = \frac{3}{4}, f = \frac{1}{10}$
				4,3 10

Where:

$$\begin{pmatrix} 1 & -1/17 & -49/51 & -7/17 \\ -1/17 & 1 & 1/3 & 1/7 \\ -49/51 & 1/3 & 1 & 3/7 \\ -7/17 & 1/7 & 3/7 & 1 \end{pmatrix}$$



Complexity bounds

Let $f_1, \ldots, f_r \in \mathbb{Z}[t_1, \ldots, t_k]$ be integer polynomials in finitely many variables. We consider a system of polynomial constraints " $f_i \bowtie 0$ " where $\bowtie \in \{=, \neq, <, \leq, \geq, >\}$.

Theorem (Tarski's transfer principle)

If a polynomial system $\{f_i \bowtie 0\}$ has a solution over \mathbb{R} , then it has a solution in a finite real extension of \mathbb{Q} .

Theorem (Real Nullstellensatz)

A polynomial F vanishes on the semialgebraic set $\mathcal{K} = \{f_i \bowtie 0\}$ if and only if $F \in \sqrt[\mathbb{R}]{\mathcal{I}(f_i \bowtie 0)}$. The ideal $\mathcal{I}(f_i \bowtie 0)$, its real radical and the membership of F can be computed.

Keyword for this decision problem: "existential theory of the reals".



Main results

Theorem

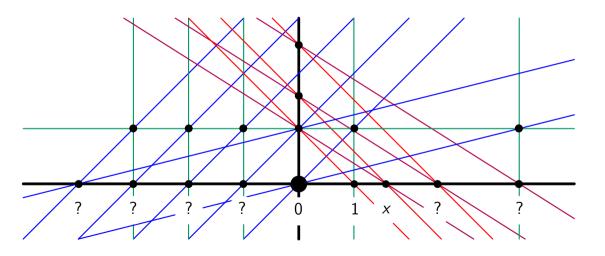
Let $d \ge 1$ and $\mathbb{Q}^{(d)}$ the field generated by all real algebraic numbers of degree at most d. For every d there exists a non-empty Gaussian CI model which has no $\mathbb{Q}^{(d)}$ -rational point.

Theorem

For every system of polynomials defining a semialgebraic set $\mathcal{K} = \{f_i \bowtie 0\}$ there exists a Gaussian CI model which is inhabited over \mathbb{R} if and only if \mathcal{K} is non-empty. Moreover, the description of this model is polynomially-sized in the description of \mathcal{K} .



Breakout riddle

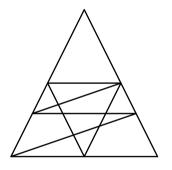


Algebra ⊆ **Synthetic geometry**

Point and line configuration for the equation $x^2 - 2 = 0$.

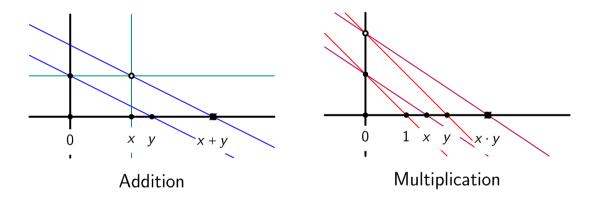
The configuration is specified by incidences between points and lines and also the parallelities of lines.

It is realizable over $\mathbb{Q}(\sqrt{2})$ but not over \mathbb{Q} .





Von Staudt constructions

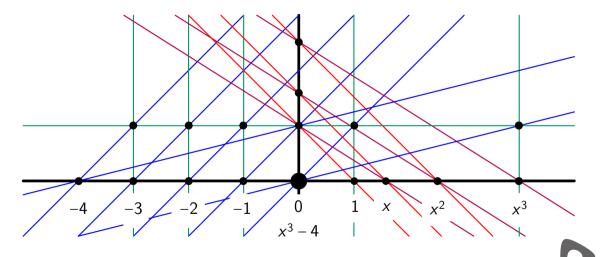


Lemma

The evaluation of integer polynomials can be encoded with incidence geometry.



The cube root of 4



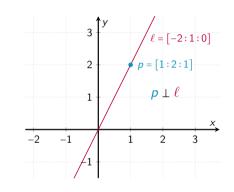
Incidence relations as conditional independence

Suppose $E = \{x, y, z\}$ and Σ_E is the identity matrix.

$$\Sigma[ij|E] = \Sigma[E] \left(\Sigma_{ij} - \Sigma_{i,E} \Sigma_{E}^{-1} \Sigma_{E,j} \right) \tag{4}$$

$$\begin{split} & \Sigma[ij|E] = 0 \iff \Sigma_{ij} = \langle \Sigma_{i,E}, \Sigma_{j,E} \rangle \\ & \Sigma[ij|] = 0 \iff \Sigma_{ij} = 0 \end{split} \iff \Sigma_{i,E} \perp \Sigma_{j,E}$$

 $p = \sum_{i,E} = [p_x : p_y : p_z]$ and $\ell = \sum_{j,E} = [\ell_x : \ell_y : \ell_z]$ are the *homogeneous coordinates* of a point and a line in the projective plane with $p \in \ell^{\perp}$.



Lemma

Incidence geometry can be encoded in CI constraints.



Condensed almost-principal minor

$$\Sigma[ij|xyz] = x_{ij}x_{xx}x_{yy}x_{zz} + x_{iz}x_{jz}\frac{x_{2y}^{2} - x_{iz}x_{jy}x_{xy}x_{xz} - x_{iy}x_{jz}x_{xy}x_{xz} + x_{iy}x_{jy}\frac{x_{2z}^{2}}{x_{xz}}x_{yy} + x_{iz}x_{jx}x_{xy}x_{yz} + x_{ix}x_{jz}x_{xz}x_{yy} - x_{ij}x_{xz}^{2}x_{yy}$$

$$+ x_{iz}x_{jy}x_{xx}x_{yz} + x_{iy}x_{jz}x_{xx}x_{yz} - x_{iz}x_{jx}x_{xy}x_{yz} - x_{ix}x_{jz}x_{xy}x_{yz}$$

$$- x_{iy}x_{jx}x_{xz}x_{yz} - x_{ix}x_{jy}x_{xz}x_{yz} + 2x_{ij}x_{xy}x_{xz}x_{yz} + x_{ix}x_{jx}x_{yz}^{2}$$

$$- x_{ij}x_{xx}x_{yz}^{2} - x_{iy}x_{jy}x_{xx}x_{zz} + x_{iy}x_{jx}x_{xy}x_{zz} + x_{ix}x_{jy}x_{xy}x_{zz}$$

$$- x_{ij}x_{xy}^{2}x_{zz} - x_{ix}x_{jx}x_{yy}x_{zz}$$

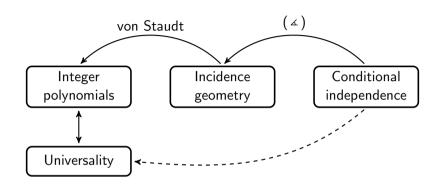
$$= x_{ij} - \sum_{k=x,y,z} x_{ik}x_{jk} = x_{ij} - \langle p_{i}, \ell_{j} \rangle.$$



Polynomial evaluation as conditional independence

	p_1		p_n	I_1		I_m	×	У	z
p_1	p_1^*		$\langle p,p' \rangle$				p_1^{\times}	$ ho_1^y$	p_1^z
÷		٠.			$\langle p,\ell angle$			÷	
p_n	$\langle p',p \rangle$		p_n^*				p_n^{\times}	p_n^y	p_n^z
I_1				ℓ_1^*		$\langle \ell, \ell' \rangle$	ℓ_1^{x}	ℓ_1^y	ℓ_1^z
:		$\langle \ell, p \rangle$			٠.			÷	
I_m				$\langle \ell', \ell \rangle$		$\ell_{m m}^*$	ℓ_{m}^{x}	ℓ_m^y	ℓ_m^z
×	$p_1^{\scriptscriptstyle X}$		p_n^{x}	ℓ_1^{x}		ℓ_{m}^{\times}	1	0	0
У	p_1^y	•••	p_n^y	ℓ_1^y	•••	ℓ_{m}^{y}	0	1	0
z	p_1^z		p_n^z	ℓ_1^z		ℓ_{m}^{z}	0	0	1 /





Theorem

To every polynomial system $\{f_i \bowtie 0\}$ there is a set of CI constraints which has a model over a field \mathbb{K}/\mathbb{Q} if and only if the polynomial system has a solution in \mathbb{K} .



QED

Theorem

Let $d \ge 1$ and $\mathbb{Q}^{(d)}$ the field generated by all real algebraic numbers of degree at most d. For every d there exists a non-empty Gaussian CI model which has no $\mathbb{Q}^{(d)}$ -rational point.

Theorem

For every system of polynomials defining a semialgebraic set $\mathcal{K} = \{f_i \bowtie 0\}$ there exists a Gaussian CI model which is inhabited over \mathbb{R} if and only if \mathcal{K} is non-empty. Moreover, the description of this model is polynomially-sized in the description of \mathcal{K} .

Question: What is the smallest $n \ge 5$ for which there is an n-variate Gaussian CI model without rational point?



Tobias Boege.

Incidence geometry in the projective plane via almost-principal minors of symmetric matrices, 2021.

arXiv:2103.02589.

Jürgen Bokowski and Bernd Sturmfels.

Computational synthetic geometry, volume 1355 of Lecture Notes in Mathematics.

Springer, 1989.

Jürgen Richter-Gebert.

Perspectives on projective geometry. A guided tour through real and complex geometry.

Springer, 2011.

Petr Šimeček.

Gaussian representation of independence models over four random variables. In *COMPSTAT conference*, 2006.



$\bowtie \in \{=\}$ suffices

In the real numbers, the solvability of a system of equations is just as hard as equations, inequations and inequalities if we introduce a new variable y:

- $f(x) \neq 0 \Leftrightarrow \exists y : yf(x) 1 = 0$
- $f(x) \ge 0 \iff \exists y : f(x) y^2 = 0$
- $f(x) > 0 \iff \exists y : y^2 f(x) 1 = 0$

(f(x)) has a multiplicative inverse) (f(x)) is a square, i.e. non-negative) (f(x)) has an inverse which is a square)

