Colorful linear structural equation models

Tobias Boege, Kaie Kubjas, Pratik Misra, Liam Solus

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Department of Mathematics and Statistics
UiT The Arctic University of Norway

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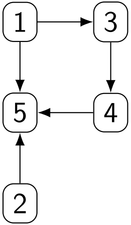
$$X_j = \sum_{i \in \mathrm{pa}(j)} \lambda_{ij} X_i + \varepsilon_j, \quad \varepsilon_j \sim \mathcal{N}(0, \omega_j).$$



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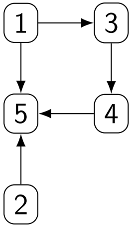

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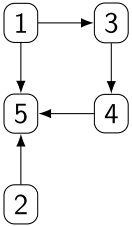

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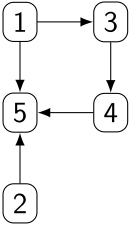

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- All such matrices form the model $\mathcal{M}(G) \subseteq \mathrm{PD}_V$.

Reasoning with graphical models

AGU PUBLICATIONS

Water Resources Research

RESEARCH ARTICLE

10.1002/2017WR020412

Key Points:

- We develop a statistical graphical model to characterize the statewide California reservoir system
- We quantify the influence of external physical and economic factors (e.g., statewide PDSI and consumer price index) on the reservoir network
- Further analysis gives a system-wide health diagnosis as a function of PDSI, indicating when heavy management practices may be needed

Supporting Information:

- Supporting Information S1
- Supporting Information S2

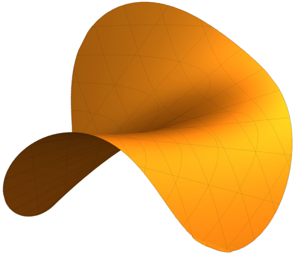
A Statistical Graphical Model of the California Reservoir System

A. Taeb[1], J. T. Reager[2], M. Turmon[2], and V. Chandrasekaran[3]

[1]Department of Electrical Engineering, California Institute of Technology, Pasadena, CA, USA, [2]Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA, USA, [3]Department of Computing and Mathematical Sciences and Department of Electrical Engineering, California Institute of Technology, Pasadena, CA, USA

Abstract The recent California drought has highlighted the potential vulnerability of the state's water management infrastructure to multiyear dry intervals. Due to the high complexity of the network, dynamic storage changes in California reservoirs on a state-wide scale have previously been difficult to model using either traditional statistical or physical approaches. Indeed, although there is a significant line of research on exploring models for single (or a small number of) reservoirs, these approaches are not amenable to a system-wide modeling of the California reservoir network due to the spatial and hydrological heterogeneities of the system. In this work, we develop a state-wide statistical graphical model to characterize the dependencies among a collection of 55 major California reservoirs across the state; this model is defined with respect to a graph in which the nodes index reservoirs and the edges specify the

Statistical models are algebraic varieties*



This set of 3-variate Gaussian distributions in a certain graphical model is defined by a single polynomial equation in its covariance matrix

$$\sigma_{12}(1 + \sigma_{23} - \sigma_{12}^2) = \sigma_{13}.$$

Algebraic statistics studies statistical models and problems using methods of algebraic geometry and computer algebra.

*sometimes

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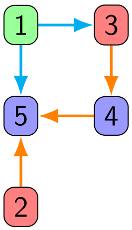
  - Markov equivalence = ambiguity about the direction of causality.

Colored Gaussian DAG models

- In a colored Gaussian DAG model, the vertices and edges of G are partitioned into color classes via a coloring function $c : V \sqcup E \to C_V \sqcup C_E$.



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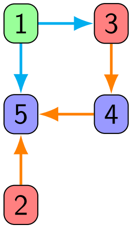

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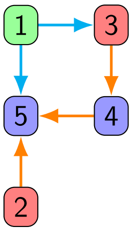

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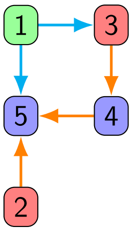
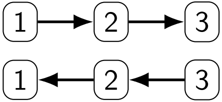
- Vertex-only colorings correspond to partial homoscedasticity [WD23].



Coloring can disambiguate the causal structure

- Coloring reduces Markov-equivalence classes which eases causal discovery.



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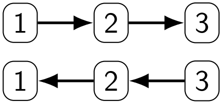

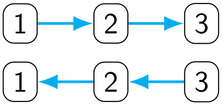

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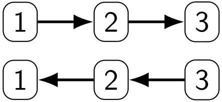

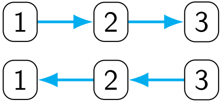

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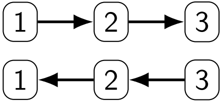

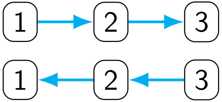

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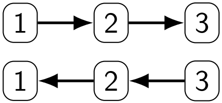

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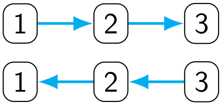

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- Not invariant anymore.

Parameter identifiability revisited

Consider the functions $\omega_{j|A}(\Sigma) = \mathrm{Var}(X_i \mid X_A)$ and $\lambda_{ij|A}(\Sigma) = \frac{\mathrm{Cov}(X_i, X_j | X_{A \setminus i})}{\mathrm{Var}(X_i | X_{A \setminus i})}$.

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This expresses the causal and coloring conditions $\lambda_{ij} = 0$, $\omega_i = \omega_j$ and $\lambda_{ij} = \lambda_{kl}$ as polynomial conditions cir, vcr and ecr in Σ.

Model geometry

Theorem

For every colored DAG (G, c) *the model* $\mathcal{M}(G, c)$ *has irreducible Zariski closure and is a smooth submanifold of* PD_V*. It is diffeomorphic to an open ball of dimension* $\mathrm{vc} + \mathrm{ec}$ *(the number of vertex- and edge-color classes).*

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- $I_c = \langle \mathrm{vcr}(i|\mathrm{pa}(i), j|\mathrm{pa}(j)) : c(i) = c(j) \rangle + \langle \mathrm{ecr}(ij|\mathrm{pa}(j), kl|\mathrm{pa}(l)) : c(ij) = c(kl) \rangle$ *is the coloring ideal of G,*
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- Resolves the colored generalization of a conjecture of Sullivant; see also [RP14].

Application to implicitization

```
needsPackage "GraphicalModels";

V = {0,1,2,3,4,5};
G = digraph(V,{{0,2},{0,3},{1,2},{1,3},{2,3},{3,4},{0,5},{1,5},{2,5},{3,5},{4,5}});
R = gaussianRing G; S = covarianceMatrix R;
allE = set(flatten for i in 0..#V-1 list for j in i+1..#V-1 list (V#i,V#j));

-- Vanishing ideal via built-in elimination method: not finished after 20 minutes
time I1 = gaussianVanishingIdeal R;

-- Vanishing ideal via saturation: 0.186855 seconds
time (
  prs = for i in V list (
    P := toList parents(G, i);
    if #P == 0 then 1 else det submatrix(S, P, P)
  );
  J = ideal for ij in toList(allE-set(edges G)) list (
    P := toList parents(G, ij#1);
    det submatrix(S, {ij#0}|P, {ij#1}|P)
  );
  I2 = fold(saturate, J, prs);
);
```

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Theorem ([WD23; STD10])

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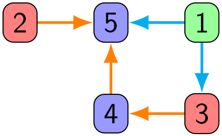
- The example on the right colors vertices and edges.
  The generic matrix in the model satisfies $X_1 \perp\!\!\!\perp X_4 \mid X_5$.
  Not faithful to G!



Structure identifiability

Theorem ([WD23])

If (G, c) and (H, c) are vertex-colored DAGs, then $\mathcal{M}(G, c) = \mathcal{M}(H, c)$ if and only if G and H are Markov-equivalent and $\mathrm{pa}_G(j) = \mathrm{pa}_H(j)$ for all $j \in V$ with $|c^{-1}(j)| \geq 2$.

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An edge-colored DAG (G, c) is BPEC if:

Structure identifiability

Theorem ([WD23])

If (G, c) and (H, c) are vertex-colored DAGs, then $\mathcal{M}(G, c) = \mathcal{M}(H, c)$ if and only if G and H are Markov-equivalent and $\mathrm{pa}_G(j) = \mathrm{pa}_H(j)$ for all $j \in V$ with $|c^{-1}(j)| \geq 2$.

An edge-colored DAG (G, c) is BPEC if:

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An edge-colored DAG (G, c) is BPEC if:

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An edge-colored DAG (G, c) is BPEC if:

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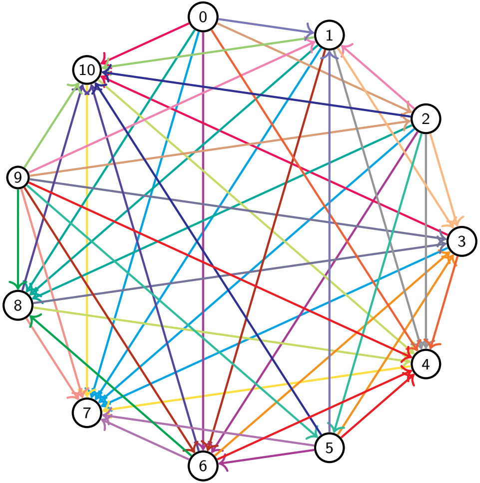
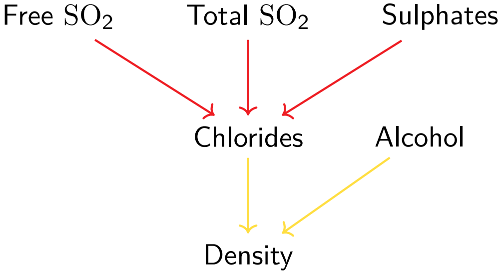
Theorem

If (G, c) and (H, d) are two BPEC-DAGs, then $\mathcal{M}(G, c) = \mathcal{M}(H, d)$ implies $(G, c) = (H, d)$. In particular, the Markov-equivalence classes of BPEC-DAGs are singletons and the causal structure is identifiable.

Wine tasting



A sensible subgraph:



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