WISST! Galois connections

A *partial order* on a set X is a relation $\leq$ which satisfies

$$x \leq x, \qquad \text{(Reflexivity)}$$
$$x \leq x' \wedge x' \leq x'' \Rightarrow x \leq x'', \qquad \text{(Transitivity)}$$
$$x \leq x' \wedge x' \leq x \Rightarrow x = x'. \qquad \text{(Antisymmetry)}$$

A partially ordered set in which each subset A has a *least upper bound* $\bigvee A$ and a *greatest lower bound* $\bigwedge A$ is a *complete lattice*.

Galois connections

A *closure operator* on $(X, \leq)$ is a function $\mathfrak{c} : X \to X$ with

$$\mathfrak{c}(x) \geq x, \qquad \text{(Extensivity)}$$
$$x \leq x' \Rightarrow \mathfrak{c}(x) \leq \mathfrak{c}(x'), \qquad \text{(Monotonicity)}$$
$$\mathfrak{c}(\mathfrak{c}(x)) = \mathfrak{c}(x). \qquad \text{(Idempotency)}$$

A *Galois connection* between $(X, \leq)$ and $(Y, \leq)$ is a pair of antitone maps $\alpha : X \to Y$ and $\beta : Y \to X$ such that $\beta\alpha$ and $\alpha\beta$ are closure operators on X and Y, respectively.

- Every topology is defined by a closure operator.
- The closed sets form a complete lattice.
- Between the closed sets of X and Y, the Galois connection establishes a *lattice antiisomorphism*.

Formal concept analysis

Let $\mathcal{O}$ be a set of *objects* and $\mathcal{A}$ a set of *attributes* with an *incidence* relation $\diamond$. This defines a Galois connection between the powersets of $\mathcal{O}$ and $\mathcal{A}$:

$$\alpha(O) := \{\, a \in \mathcal{A} : o \diamond a \;\; \forall o \in O \,\},$$
$$\beta(A) := \{\, o \in \mathcal{O} : o \diamond a \;\; \forall a \in A \,\}.$$

The closure operator $\beta\alpha$ *saturates* a set of objects O with respect to all of its attributes.

The other closure operator $\alpha\beta$ *infers* all attributes which are implied on $\mathcal{O}$ by a set of attributes A.

Example: Galois theory

Let L/F be a finite field extension with $G = \mathrm{Aut}(L/F)$.
Then the maps

$$L/K/F \quad \longleftrightarrow \quad 1 \leq H \leq G,$$

given by “fixgroup” and “fixed field” are a Galois connection between intermediate fields in L/F and subgroups of G.

If L/F is Galois, then the Fundamental theorem of Galois theory shows that every intermediate field and every subgroup is closed with respect to the connection.

This connection is defined by the relation $x \diamond g :\Leftrightarrow g(x) = x$.

Example: Algebraic geometry

For $\mathbb{K}$ algebraically closed, consider the relation

$$a \diamond f \;\;:\Leftrightarrow\;\; f(a) = 0$$

on $\mathbb{K}^n$ (objects) and $\mathbb{K}[x_1, \ldots, x_n]$ (attributes).

The closed object sets are the algebraic varieties and the closed attribute sets are the radical ideals, by Hilbert's Nullstellensatz.

Example: Convex geometry

Between $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$ define

$$x \diamond \alpha \ :\Leftrightarrow\ \alpha(x) \geq 0.$$

This gives a Galois connection whose closed object sets are the closed convex cones, by the separation lemma of convex sets.

The dual closed sets are closed convex cones as well, in $(\mathbb{R}^n)^*$. The Galois connection constructs the *dual cone* in both directions.

Example #4

Let $\mathcal{K} = \{ g_i \geq 0 \}$ be a semialgebraic set. Consider the following relation between subsets of $\mathcal{K}$ and subets of $\{ g_i \}$:

$$x \diamond g \;\; :\Leftrightarrow \;\; g(x) = 0.$$

What is the name of this construction? If $\mathcal{K}$ is a polyhedron, then its closed subsets form the *face lattice*.