On the Intersection and Composition properties for discrete random variables

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- The CI symbols are symmetric $[I \perp\!\!\!\perp J \mid K] \iff [J \perp\!\!\!\perp I \mid K]$.
- A set S of CI symbols is a semigraphoid if it satisfies

$$\begin{aligned}[I \perp\!\!\!\perp JK \mid L] &\iff [I \perp\!\!\!\perp J \mid L] \wedge [I \perp\!\!\!\perp K \mid JL] \\ &\iff [I \perp\!\!\!\perp K \mid L] \wedge [I \perp\!\!\!\perp J \mid KL]\end{aligned}$$

- E.g., conditional independence relation of every system of random variables.

Partial converses of the semigraphoid property

$$[I \perp\!\!\!\perp JK \mid L] \implies \begin{cases} \textcircled{1}[I \perp\!\!\!\perp J \mid L] \wedge \textcircled{2}[I \perp\!\!\!\perp J \mid KL] \wedge \\ \textcircled{3}[I \perp\!\!\!\perp K \mid L] \wedge \textcircled{4}[I \perp\!\!\!\perp K \mid JL] \end{cases}$$

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- $\textcircled{1} \wedge \textcircled{4}$ and $\textcircled{2} \wedge \textcircled{3}$ are sufficient for $[I \perp\!\!\!\perp JK \mid L]$ by semigraphoid axioms.
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Modulo the semigraphoid axioms Intersection and Composition are logical converses:

Intersection	② ∧ ④	$\implies$	① ∧ ③
Composition	② ∧ ④	$\impliedby$	① ∧ ③

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- Curiously they are dual to each other via $[I \perp\!\!\!\perp J \mid K]^* := [I \perp\!\!\!\perp J \mid N \setminus IJK]$:

Intersection $\quad [I \perp\!\!\!\perp \boldsymbol{J} \mid \boldsymbol{K} L] \wedge [I \perp\!\!\!\perp \boldsymbol{K} \mid \boldsymbol{J} L] \implies [I \perp\!\!\!\perp \boldsymbol{J} \mid L] \wedge [I \perp\!\!\!\perp \boldsymbol{K} \mid L]$

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\text{Intersection} & [I \perp\!\!\!\perp J \mid KL] \wedge [I \perp\!\!\!\perp K \mid JL] \implies [I \perp\!\!\!\perp J \mid L] \wedge [I \perp\!\!\!\perp K \mid L] \\
\quad\quad\downarrow & \text{use } \text{Ł} = N \setminus IJKL \\
\text{Intersection}^* & [I \perp\!\!\!\perp J \mid \text{Ł}] \wedge [I \perp\!\!\!\perp K \mid \text{Ł}] \implies [I \perp\!\!\!\perp J \mid K\text{Ł}] \wedge [I \perp\!\!\!\perp K \mid J\text{Ł}]
\end{array}$$

but this is Composition with L replaced by Ł.

Examples

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  - d-separation, u-separation, m-separation, **$*$-separation**
- Positive distributions satisfy Intersection.
- MTP_2 distributions satisfy Composition.

Intersection for three binary random variables

$$[I \perp\!\!\!\perp J \mid KL] \wedge [I \perp\!\!\!\perp K \mid JL] \implies [I \perp\!\!\!\perp J \mid L] \wedge [I \perp\!\!\!\perp K \mid L]$$

- By marginalizing to $IJKL$, conditioning on L and viewing I, J, K as single random variables, we can reduce one instance of Intersection to the trivariate case.

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$$\langle p_{110}, p_{101}, p_{010}, p_{001} \rangle \cap \langle p_{111}, p_{100}, p_{011}, p_{000} \rangle$$

- Failure of Intersection only on the boundary. Full support implies Intersection.

The common information criterion

Let g be a Gács–Körner common information of j and k, i.e., it solves the problem

$$\begin{aligned} &\max H(g) \\ &\text{ s.t. } H(g \mid j) = H(g \mid k) = 0. \end{aligned}$$

Theorem

If $[i \perp\!\!\!\perp j \mid k]$ *and* $[i \perp\!\!\!\perp k \mid j]$, *then* $[i \perp\!\!\!\perp jk \mid g]$. *Hence, if* g *is constant then* $[i \perp\!\!\!\perp jk]$.

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- Also known as the Double Markov property [CK11, Exercise 16.25].

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Conditional Ingleton vs. Gács–Körner

It is not difficult to parametrize binary distributions which satisfy the conditional Ingleton criterion but fail the common information criterion using Cylindrical Algebraic Decomposition in `Mathematica`, e.g.,

i	j	k	g	Pr
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- Distribution on ijk is quasi-uniform and $[i \perp\!\!\!\perp jk]$ holds.

Composition for three binary random variables

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There is only one irreducible component of $\mathscr{M}([i \perp\!\!\!\perp j] \wedge [i \perp\!\!\!\perp k])$ *on which the sum of all probabilities does not vanish.*

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- There exist positive distributions violating Composition.

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- This is formally dual to the conditional Ingleton criterion for Intersection.
- The Composition property is obtained conditionally on g.
- How to use this? Any constructions of suitable g?

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Děkuji!

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